WETD—A Finite Element Time–Domain Approach for Solving Maxwell's Equations

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***Abstract*— A family of finite element time-domain methods, WETD(Θ), is derived to solve the time-varying Maxwell's equations. The proposed methodology is based upon the application of the Faedo-Galerkin procedure and the use of the Whitney 1-forms as bases to result in an ordinary differential equation in time for the electric field. Moreover, the resultant ordinary differential equation is solved by employing central and/or backward difference approximations. Since the WETD methods presented here are used in conjunction with tetrahedral finite element meshes, it imposes no limitations on the problem geometry. Also, in this contribution, a general stability condition has been derived for the WETD(Θ) method of which the central and backward differences are special cases corresponding to $\Theta = 1$ and $\Theta = 0$, respectively.**

I. Introduction

THE finite difference time-domain (FDTD) [1] algorithm has been used widely in solving the transient responses of electromagnetic problems. However, in its original form, it is difficult to model complex EM problems with curved surfaces using the FDTD method. Many variants have been proposed in the past with the aim to circumvent this difficulty with varying degrees of success. Almost all of these approaches are based upon, one form or the other, the use of finite difference approximation in both spatial and temporal domains. It is the purpose of this paper to show a finite element time domain formulation, Whitney element time domain (WETD) method, which uses Whitney 1-forms in the spatial domain and the finite difference in the time domain, respectively, for solving Maxwell's equations. In this way, the proposed WETD method can be used on a tetrahedral finite element mesh and consequently, it imposes no geometric limitations. Furthermore, in this contribution, we also generalize the formulation by the Θ method, will be described later, to a family of WETD methods depending on the Θ value. Therefore, the time step can always be set with compatible resolution with its spatial counterparts, i.e. $c\delta t \approx \Delta x$, by choosing a suitable Θ.

II. Faedo–Galerkin Formulation

In this paper, we consider the solution of Maxwell's equations in space-time R^4

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}. \tag{1}$$

From these equations, an initial value problem can be derived in terms of the electric field $\vec{E}$ as

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{E} + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{\partial \vec{J}}{\partial t} \quad in\ \Omega$$
$$\hat{n} \times \vec{E} = 0 \quad on\ \Gamma_e$$
$$\hat{n} \times \nabla \times \vec{E} = 0 \quad on\ \Gamma_h. \tag{2}$$

The weak form, Galerkin form, of (2) is just

$$\int_\Omega \left(\vec{v} \bullet \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \vec{v} \bullet \nabla \times \frac{1}{\mu} \nabla \times \vec{E} \right) d\Omega = -\int_\Omega \vec{v} \bullet \frac{\partial \vec{J}}{\partial t} d\Omega, \tag{3}$$

where $\vec{v}$ is a test function. To achieve a greater symmetry between trial and test vector functions, we apply vector identities to result in

$$\int_\Omega \left[\vec{v} \bullet \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{\mu} (\nabla \times \vec{v}) \bullet \left(\nabla \times \vec{E} \right) \right] d\Omega = -\int_\Omega \vec{v} \bullet \frac{\partial \vec{J}}{\partial t} d\Omega. \tag{4}$$

Consequently, the application of the Faedo-Galerkin procedure can be stated as following. Given an N-dimensional subspace $S^h \subset \mathcal{H}^1$, where $\mathcal{H}^1 = \{\vec{v} \mid \vec{v}, \partial\vec{v} \in (L^2)^3\}$, the Galerkin principle is to find a vector function $\vec{E}^h(\vec{r}; t)$ with the following property: At each $t > 0$, $\vec{E}^h$ lies in S^h and satisfies

$$\int_\Omega \left[\vec{v}^h \bullet \epsilon \frac{\partial^2 \vec{E}^h}{\partial t^2} + \frac{1}{\mu} (\nabla \times \vec{v}^h) \bullet \left(\nabla \times \vec{E}^h \right) \right] d\Omega + \int_\Omega \vec{v}^h \bullet \frac{\partial \vec{J}^h}{\partial t} d\Omega = 0, \qquad \forall \vec{v}^h \in S^h. \tag{5}$$

Notice that the time variable is still continuous: The Faedo-Galerkin formulation is discrete in the spatial variables and yields a system of ordinary differential equations in time. To make this formulation operational, we use the Whitney 1-forms as the bases for the trial space S^h and expand the trial and test vector functions as

$$\vec{E}^h(\vec{r}; t) = \vec{W}_i(\vec{r})\ e^i(t) \tag{6}$$

where $e^i, \vec{W}_i$ are the circulation of the electric field and the vector basis function associated with edge i. Also, in (6) we have used the Einstein notation for the summation over index i.

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Finally, the optimal weights e^i are determined by the Galerkin principle

$$\int_\Omega \left(\epsilon \vec{W}^j \bullet \vec{W}_i\right) d\Omega \ \frac{d^2 e^i}{dt^2} + \int_\Omega \left(\frac{1}{\mu} \nabla \times \vec{W}^j \bullet \nabla \times \vec{W}_i\right) d\Omega \ e^i$$

$$= -\int_\Omega \left(\vec{W}^j \bullet \frac{\partial \vec{J}^h}{\partial t}\right) d\Omega, \qquad \forall j. \tag{7}$$

Now we put the Galerkin equation into vector notation, with $\mathcal{E}$ as the coefficient vector, the result is

$$[T]\frac{d^2\mathcal{E}}{dt^2} + c^2[S]\mathcal{E} = -\mathcal{F} \tag{8}$$

where

$$[T]_{ij} = \int_\Omega \left(\epsilon_r \vec{W}^j \bullet \vec{W}_i\right) d\Omega$$

$$[S]_{ij} = \int_\Omega \left(\frac{1}{\mu_r} \nabla \times \vec{W}^j \bullet \nabla \times \vec{W}_i\right) d\Omega \tag{9}$$

and the components of the right-hand-side are

$$\mathcal{F}_j = \int_\Omega \vec{W}^j \bullet \frac{\partial \vec{J}^h}{\partial t} d\Omega. \tag{10}$$

If there were inhomogeneous boundary conditions, time-dependent or not, their effects would also appear in $\mathcal{F}$.

III. Finite Differences in Time

The ordinary differential equation in (8) typically can be approximated by finite difference formulae in three ways: forward difference, backward difference, and central difference. The forward difference formula when applies to (8) becomes unconditionally unstable in the numerical implementation. Therefore, in this contribution, we will focus only on the central, backward differences and their combinations. Furthermore, to simplify the discussion here, we shall also assume that $\mathcal{F} = 0$, however, the inclusion of $\mathcal{F}$ in the final formulation is straightforward and will not affect the main concepts in this contribution.

A. Central Difference

By applying the central difference to the time derivative in (8), we obtain

$$[T]\frac{1}{\delta t^2}\left(\mathcal{E}^{n+1} - 2\mathcal{E}^n + \mathcal{E}^{n-1}\right) + c^2[S]\mathcal{E}^n = 0. \tag{11}$$

The above equation can be rearranged to result in a matrix equation which can be used to update the coefficient vector, $\mathcal{E}$, as

$$[T]\mathcal{E}^{n+1} = -[T]\mathcal{E}^{n-1} + \left(2[T] - c^2\delta t^2[S]\right)\mathcal{E}^n. \tag{12}$$

As evidenced in (12), the updating of the electric field, for each time step, requires solving a matrix equation of the form:

$$[T]x = y. \tag{13}$$

The stability condition of this approach will be described later as a special case in the Θ method with $\Theta = 1$.

B. Backward Difference

When the O.D.E. (8) is approximated by backward difference, the following equation is obtained:

$$[T]\frac{1}{\delta t^2}\left(\mathcal{E}^{n+1} - 2\mathcal{E}^n + \mathcal{E}^{n-1}\right) + c^2[S]\mathcal{E}^{n+1} = 0. \tag{14}$$

Finally, the matrix equation that is used to update the coefficient vector, $\mathcal{E}^{n+1}$, is written as

$$\left([T] + c^2\delta t^2[S]\right)\mathcal{E}^{n+1} = 2[T]\mathcal{E}^n - [T]\mathcal{E}^{n-1}. \tag{15}$$

Like the central difference approach, for each time step the backward difference approach also requires solving a matrix equation of the form

$$\left([T] + c^2\delta t^2[S]\right)x = y. \tag{16}$$

However, as can be seen later, the algorithm results in an unconditional-stable time marching numerical scheme.

C. Θ Method

It is well-known that the central difference will give second order accuracy in time, and whereas, the backward difference will only be accurate upto first order. A general finite difference scheme that we proposed here is based upon a concept which is similar to the Generalized Θmethod [3] for solving parabolic equations. We multiply (12 by a real number, $0 \le \Theta \le 1$, and (15) by $1 - \Theta$, and sum the two equations together to form

$$\left\{[T] + (1-\Theta)c^2\delta t^2[S]\right\}\mathcal{E}^{n+1} =$$

$$-[T]\mathcal{E}^{n-1} + \left\{2[T] - \Theta c^2\delta t^2[S]\right\}\mathcal{E}^n. \tag{17}$$

From (17), we see that $\Theta = 0$ and $\Theta = 1$ reduce to backward and central difference schemes, respectively.

D. Stability Condition

Since the central and backward differences are special cases of the Θ method, we will only derive the stability condition for the Θ method in this section. By defining a growing factor α as

$$\alpha = \frac{\| \mathcal{E}^{n+1} \|}{\| \mathcal{E}^n \|}. \tag{18}$$

Then we say that a numerical scheme is stable if and only if

$$\lim_{n\to\infty} \alpha \le 1. \tag{19}$$

From (17), it can be shown that the Θ method will be stable if and only if the following condition holds

$$\rho\left(\left\{2[T] - \Theta c^2\delta t^2[S]\right\}^2\right) \le 4\rho\left([T]\left\{[T] + (1-\Theta)c^2\delta t^2[S]\right\}\right) \tag{20}$$

where $\rho(A)$ is the spectral radius of the matrix A. Finally, with algebraic operations, the final stability condition can be deduced as

$$c\delta t \le \frac{2}{\Theta}\sqrt{\frac{\lambda^T_{\min}}{\lambda^S_{\max}}} \tag{21}$$

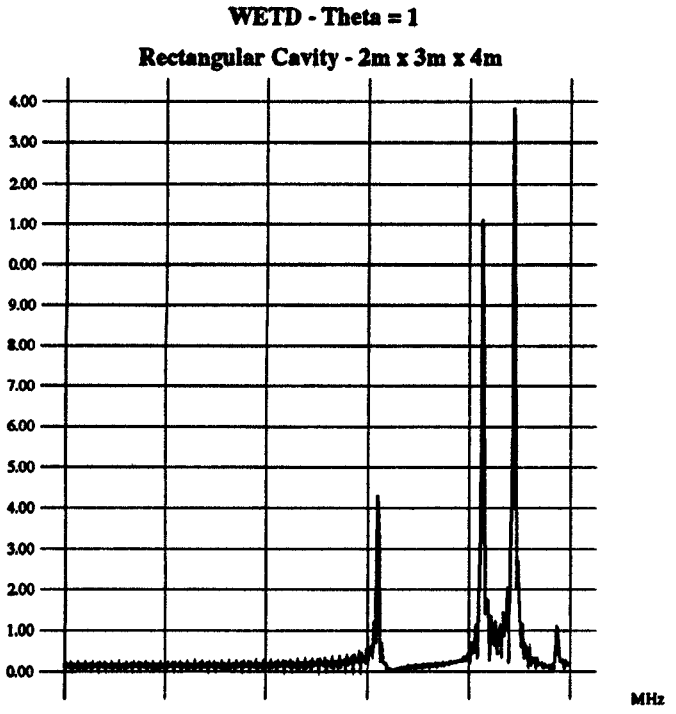

Fig. 1. The FFT of the time signal for a rectangular cavity with dimensions 2 m x 3 m x 4 m.

where $\lambda^T_{\min}$ and $\lambda^S_{\max}$ are the minimum and the maximum eigenvalues of matrices $[T]$ and $[S]$, respectively. Moreover, we have

$$\frac{\lambda^T_{\min}}{\lambda^S_{\max}} = \frac{1}{\kappa_{\max}} \tag{22}$$

where κ is the maximum eigenvalue of the following generalized eigenmatrix equation:

$$[S]x = \kappa[T]x. \tag{23}$$

In practice, (23) can be solved efficiently by using Lanczos algorithm [4].

TABLE I
RESONANT FREQUENCIES OF THE FIRST THREE MODES

Resonant frequencies (MHz)	
Exact	WETD(1.0)
62.5	62.07 407
83.853	82.59 259
90.139	88.74 074

IV. NUMERICAL RESULTS

A simple rectangular cavity with dimensions 2 m x 3 m x 4 m has been studied by using the WETD method presented in this contribution. The cavity is first discretized into tetrahedra with typical element size corresponds to $h = 0.25$ m. The numerical algorithm starts by assigning a divergence-free excitation with a Gaussian distribution of the form $\exp -((t - 5\sigma)/\sigma)^2$ for the electric field. The constant σ is chosen as $\sigma = 1/(\pi f)$ and $f = 500$ MHz. The numerical experiment has been conducted using $\Theta = 1$, and time step is calculated as

$$c\delta t = \frac{1.9}{\Theta}\sqrt{\frac{\lambda^T_{\min}}{\lambda^S_{\max}}} = 0.06209. \tag{24}$$

It took 12 035 CPU time to run for 10 000 time steps on a HP 735 workstation. The FFT of the time signal obtained at a certain observation point is shown in Fig. 1. The resonant frequencies of the first three modes compared to the exact solutions is given in Table I. Good agreements are obtained.

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